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Substituting this value, the result follows by a simple algebraic reduction. The result is true only when the series converges, *i. e.*, when $|(x^2-1)(z/(1-xz))^2|<1$ or $x<(1+z^2)/2z$.

Also solved by S. A. JOFFE, GEO. W. HARTWELL, ALBERT N. NAUER, and A. M. HARDING. *Note.*—This problem should have been placed in the Algebra department. EDITORS.

**377. Proposed by W. D. CAIRNS, Oberlin College.**

It is required to find a curve of the form $y = x(x-a)(x-b)$ such that the abscissas of the maximum and minimum values, as well as $a$ and $b$, shall be positive integers.

### SOLUTION BY RALPH D. BEETLE, Dartmouth College.

The abscissas of the minimum and maximum values are

$$(1)\qquad x_1 = \tfrac{1}{3}[a+b+\sqrt{a^2-ab+b^2}],\qquad x_2 = \tfrac{1}{3}[a+b-\sqrt{a^2-ab+b^2}].$$

If $a$ and $b$ are positive integers, so also are $x_1$ and $x_2$ if, and only if, $a+b$ is divisible by 3 and $a^2-ab+b^2$ is a perfect square divisible by 9. These conditions are equivalent to the simpler ones that $a$ and $b$ be divisible by 3 and $a^2-ab+b^2$ be a perfect square. If we write

$$(2)\qquad a = 3h,\ b = 3k$$

our problem is reduced to the finding of positive integral values of $h$ and $k$ such that $h^2-hk+k^2$ is a perfect square. We note at once that, if $a$ and $b$ are equal, it is sufficient that they be multiples of 3. We may find all solutions of the problem as follows:

Without loss of generality, we may assume that $h \leqq k$, so that $h^2-hk+k^2 \leqq k^2$. If we put $h^2-hk+k^2=(k-r)^2$, it is found that

$$(3)\qquad h = \tfrac{1}{2}[k \pm \sqrt{k^2-8rk+4r^2}].$$

In order that $h$ be an integer, we must have $k^2-8rk+4r^2=t^2$, where $t$ is an integer, or $(k-4r)^2-t^2=12r^2$. It is evident that $k$ and $t$ are both even or both odd, and we may therefore put

$$(4)\qquad k-4r=n+s,\qquad t=n-s,$$

where $n$ and $s$ are positive integers. Then

$$(5)\qquad ns=3r^2,$$

and the values of $a$, $b$, $x_1$, $x_2$ in terms of $r$, $n$, $s$ are found from (1), (2), (3), (4) to be either

$$a=3n+6r\qquad b=3n+12r+3s,\qquad x_1=3n+9r+2s,\qquad x_2=n+3r,$$

or

$$a=6r+3s,\qquad b=3n+12r+3s,\qquad x_1=2n+9r+3s,\qquad x_2=3r+s.$$

Since the two sets differ only in the interchange of $n$ and $s$, only one set is needed in view of the symmetric roles of $n$ and $s$ in the relation (5). We may therefore state our result as follows:

If $n$, $s$ and $r$ are integers such that $n > 0$, $s \geqq 0$, $r \geqq 0$, $ns = 3r^2$, and if $a = 3n + 6r$, $b = 3n + 12r + 3s$, the abscissas of the minimum and maximum points of the curve, $y = x(x - a)(x - b)$, are the positive integers $x_1 = 3n + 9r + 2s$ and $x_2 = n + 3r$.

The smallest unequal values of $a$ and $b$ are 9 and 24. In the following table are the solutions which correspond to $r = 1$ and $r = 2$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $n$ | 1 | 3 | 1 | 2 | 3 | 4 | 6 | 12 |
| $s$ | 3 | 1 | 12 | 6 | 4 | 3 | 2 | 1 |
| $r$ | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 |
| $a$ | 9 | 15 | 15 | 18 | 21 | 24 | 30 | 48 |
| $b$ | 24 | 24 | 63 | 48 | 45 | 45 | 48 | 63 |
| $x_1$ | 18 | 20 | 45 | 36 | 35 | 36 | 40 | 56 |
| $x_2$ | 4 | 6 | 7 | 8 | 9 | 10 | 12 | 18 |

The consideration of negative values of $n$ and $s$ is unnecessary, since it can be proved that no additional solutions of the problem are thus obtained.

Also solved by ALBERT N. NAUER, W. C. EELLS, JOSEPH B. REYNOLDS, ELMER SCHUYLER ELIJAH SWIFT, HORACE OLSON, and the PROPOSER.

## MECHANICS.

**292. Proposed by C. N. SCHMALL, New York City.**

In a bombardment, a battleship directs its fire at a fort standing on a hill whose height is $a$ feet above the sea level. The angle of elevation of the fort is found to be $\phi$. If the initial velocity of the projectile is $v$, show that the fort will *not* be struck if $v < \sqrt{ag(1 + \csc\phi)}$.

### SOLUTION BY H. S. UHLER, Yale University.

Let the coördinate plane contain the fort and the vertical through the ship. Also let $\theta$ denote the angle which the rifle makes with the positive (upward) direction of the axis of $y$. At any point of the trajectory

$$x = v \sin\theta \cdot t,$$

$$y = v \cos\theta \cdot t - \tfrac{1}{2}gt^2,$$

hence

$$gx^2 - v^2 \sin 2\theta \cdot x + 2v^2 \sin^2\theta \cdot y = 0.$$

This parabola will pass through the fort ($x = a \cot\phi$, $y = a$) when

$$a^2 g \cot^2\phi \cdot \cot^2\theta - 2av^2 \cot\phi \cdot \cot\theta + a(2v^2 + ag \cot^2\phi) = 0.$$

The roots of this quadratic in $\cot\theta$ will be complex if

$$(v^2 - ag)^2 - (ag \csc\phi)^2$$

is negative. Since this expression may be written

$$[v^2 + ag(\csc\phi - 1)] \cdot [v^2 - ag(\csc\phi + 1)],$$